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Subsingular vectors and conditionally invariant (q-deformed) equations

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Abstract. We give a systematic discussion of the relation between subsingular vectors of Verma modules over semisimple Lie algebras $\mathcal{G}$ and differential equations which are conditionally $\mathcal{G}$-invariant. This is extended to the Drinfeld–Jimbo q-deformation $U_q(\mathcal{G})$ of $\mathcal{G}$. We treat in detail the conformal algebra $su(2,2)$, its complexification $sl(4)$ and their q-deformations. The conditionally invariant equations are the d'Alembert equation and a new equation arising from a subsingular vector proposed by Bernstein–Gel'fand–Gel'fand. We also give the q-difference analogues of these equations.

1. Introduction

It is well known that the d'Alembert equation

$$\Box\, f(x) = 0 \tag{1}$$

is Poincaré and even conformal invariant, see [1], for example. Here $f(x)$ is a scalar field of fixed conformal weight, $x = (x_0, x_1, x_2, x_3)$ denotes the Minkowski space–time coordinates, and $\Box$ is the d'Alembert operator $\Box = \partial^\mu \partial_\mu = (\vec{\partial})^2 - (\partial_0)^2$.

In this paper we would like to present representation-theoretic reinterpretations of this fact. There are two aspects to this. First, from the point of view of induced representations of the conformal algebra $su(2,2)$ one cannot automatically obtain representations which are also irreducible finite-dimensional (e.g., scalar above) representations of the Lorentz subalgebra. To ensure this one has to impose additional conditions and to restrict oneself to functions which obey these conditions. In the case at hand there are two such conditions and it is on such functions that (1) is conformal invariant. That is why we shall call the conformal invariance of (1) *conditional*. (Using approaches different from ours other conditionally invariant equations were considered in [2–6], (for further comment see subsection 3.1).)

The second aspect is that we can find a counterpart of (1) in the representation theory of Verma modules over the complexification $sl(4)$ of $su(2,2)$. Namely, this counterpart is a *subsingular* vector of a Verma module (definition below).

In this paper we consider (1) and conditionally invariant equations in general applying the approach of [7]. The required results from [7] stated in condensed form (given in some detail in subsection 3.1) are: to every *singular* (*subsingular*) vector of a Verma module over a semi-simple, and also reductive, Lie algebra $\mathcal{G}$ there corresponds a differential operator

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and equation *invariant* (*conditionally invariant*) with respect to $\mathcal{G}$. (Both statements are also valid for the corresponding Lie group with some additional subtleties, see [7].)

One of the specifics of the approach of [7] is that if one wants to consider (conditional) invariance with respect to some real Lie algebra $\mathcal{G}_0$ one has also to know the invariance with respect to the complexification $\mathcal{G}$ of $\mathcal{G}_0$. The same is true in the q-deformed case. That is why we treat $sl(4)$ and the conformal algebra $su(2,2)$ in parallel, and analogously $U_q(sl(4))$ and $U_q(su(2,2))$.

We treat the $q=1$ case in detail since some of our results are also new in this classical situation. In particular, we also give (1) with a non-trivial right-hand side and we present a new conditionally invariant equation.

The paper is organized as follows. The notion of subsingular vector is explained in subsection 2.1 for arbitrary (q-deformed) simple Lie algebras. Then we restrict ourselves to $sl(4)$ and $U_q(sl(4))$ and we give the singular and subsingular vectors we shall need. In parallel we give the explicit conditions for irreducibility of the lowest-weight modules. Here the exposition is common for generic q. These results are applied respectively in sections 3 and 4 to obtain the conditionally invariant equations for $q=1$ (see equations (50), (57)) and for generic q (see equations (71), (73)) explicitly (given together with the equations ensuring their invariance).

2. Subsingular vectors

2.1. Let $\mathcal{G}=\mathcal{G}^+\oplus\mathcal{H}\oplus\mathcal{G}^-$ be a semisimple Lie algebra, where $\mathcal{H}$ is a Cartan subalgebra of $\mathcal{G}$, $\mathcal{G}^+$ ($\mathcal{G}^-$) are the positive (negative) root vector spaces of the root system $\Delta=\Delta(\mathcal{G},\mathcal{H})$, corresponding to the decomposition $\Delta=\Delta^+\cup\Delta^-$ into positive and negative roots. Let $\Delta_S=\{\alpha_i \mid i=1,\ldots,r=\operatorname{rank}\mathcal{G}\}$ be the system of simple roots of Δ. We use the standard deformation $U_q(\mathcal{G})$ [8,9] given in terms of the Chevalley generators $X_i^\pm$, $H_i\in\mathcal{H}$, $i=1,\ldots,r$ of $\mathcal{G}$. (The explicit relations we give in appendix A for $\mathcal{G}=sl(4)$. For general $\mathcal{G}$ see [8,9], or in the same notation as here [10].) The elements H_i span the Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}$, while the elements $X_i^\pm$ generate the subalgebras $U_q(\mathcal{G}^\pm)$.

A lowest-weight module (LWM) M^Λ over $U_q(\mathcal{G})$ is given by the lowest weight $\Lambda\in\mathcal{H}^*$ ($\mathcal{H}^*$ is the dual of $\mathcal{H}$) and a lowest-weight vector v_0 so that $Xv_0=0$ if $X\in\mathcal{G}^-$, $Hv_0=\Lambda(H)v_0$ if $H\in\mathcal{H}$. In particular, we use the Verma modules V^Λ over $U_q(sl(\mathcal{G}))$ which are the lowest-weight modules such that $V^\Lambda\cong U_q(\mathcal{G}^+)v_0$.

Let us introduce the numbers

$$m_i=m_i(\Lambda)\doteq(\rho-\Lambda)(H_i)=1-\Lambda(H_i)=1-(\Lambda,\alpha_i^\vee)\qquad i=1,\ldots,r \tag{2}$$

where $\rho=\frac{1}{2}\sum_{\beta\in\Delta^+}\beta$, ($\rho(H_k)=(\rho,\alpha_k)=1$) and $(\cdot,\cdot)$ is the scalar product of the roots normalized so that for the short roots α we have $(\alpha,\alpha)=2$, $\alpha^\vee\equiv 2\alpha/(\alpha,\alpha)$.

We note that these numbers completely determine the lowest weight Λ and will also be used for the characterization of the LWM. The collection of these numbers will be called the *signature* of Λ and denoted $\chi(\Lambda)$ or just χ:

$$\chi=\chi(\Lambda)\doteq(m_1,\ldots,m_r)\,. \tag{3}$$

Analogously, we shall also use numbers corresponding to arbitrary positive roots

$$m_\alpha=m_\alpha(\Lambda)\doteq(\rho-\Lambda)(H_\alpha)=(\rho-\Lambda,\alpha^\vee)\qquad \alpha\in\Delta^+ \tag{4}$$

where $H_\alpha\in\mathcal{H}$ corresponds to the root α by the isomorphism $\mathcal{H}\cong\mathcal{H}^*$, (as H_i corresponds to α_i). Certainly, each m_α is a fixed linear combination of m_i, however, these numbers have independent importance as we shall see just below. Naturally, $m_{\alpha_i}=m_i$.

In this paper we restrict ourselves to the case when the deformation parameter q is not a non-trivial root of 1. (For the case where q is a non-trivial root of 1 we refer the reader to [10].) In this case a Verma module V^Λ is reducible [11] ($q = 1$), [10] iff at least one of the numbers m_α is a positive integer:

$$m_\alpha \in \mathbb{N} \tag{5}$$

Whenever (5) is fulfilled there exists a *singular* vector $v_s = v^{\alpha,m_\alpha}$ in V^Λ such that $v_s \notin \mathbb{C}v_0$, $Xv_s = 0$, $\forall\ X \in \mathcal{G}^-$ and $Hv_s = (\Lambda + m_\alpha\alpha)(H)\,v_s$, $\forall\ H \in \mathcal{H}$. The space $I^\alpha = U_q(\mathcal{G}^+)\,v^{\alpha,m_\alpha}$ is a proper submodule of V^Λ isomorphic to the Verma module $V^{\Lambda+m_\alpha\alpha}$ with a shifted lowest weight $\Lambda + m_\alpha\alpha$ [7, 10]. Clearly, this implies that V^Λ and $V^{\Lambda+m_\alpha\alpha}$ have the same values of the Casimir operators.

Remark 1. Note that if we were considering *highest* instead of lowest weights, the analogue of the numbers m_k, (m_α), would be defined as $m_k^{\rm hw} = 1+\Lambda(H_i)$, ($m_\alpha^{\rm hw} = (\rho+\Lambda)(H_\alpha)$); the shifted weight is $\Lambda - m_\alpha^{\rm hw}\alpha$. However, the statement about reducibility is unchanged [7].

It is important that one should be able to find explicit formulae for the singular vectors. The singular vector introduced above is given by [7, 10]

$$v_s = v^{\alpha,m_\alpha} = \mathcal{P}^{\alpha,m_\alpha}(X_1^+, \ldots, X_r^+)\ v_0 \tag{6}$$

where $\mathcal{P}^{\alpha,m_\alpha}$ is a homogeneous polynomial in its variables of degrees mn_i, where $n_i \in \mathbb{Z}_+$ come from $\alpha = \sum n_i\alpha_i$, α_i is the system of simple roots. The polynomial $\mathcal{P}^{\alpha,m_\alpha}$ is unique up to a non-zero multiplicative constant. The papers [7, 10] contain all the explicit singular vectors needed in this paper. Note that we refer to both, since [7] gives formulae for $q = 1$, while [10] gives such formulae for general q. (More general explicit formulae for singular vectors, including all singular vectors for $U_q(sl(n))$, are contained in [12]. Note that the modules considered in [10, 12] are highest-weight modules and the singular vectors are polynomials in X_i^-; the translation of those formulae to the lowest-weight module setting is straightforward in view of the above remark.)

Certainly, equation (5) may be fulfilled for several positive roots (even for all of them). Let Δ_Λ denote the set of all positive roots for which (5) is fulfilled, and let us denote: $\tilde{I}^\Lambda \equiv \cup_{\alpha\in\Delta_\Lambda} I^\alpha$. Clearly, $\tilde{I}^\Lambda$ is a proper submodule of V^Λ. Let us also denote $F^\Lambda \equiv V^\Lambda/\tilde{I}^\Lambda$.

Furthermore, we shall also use the following notion. The singular vector v_1 is called a *descendent* of the singular vector $v_2 \notin \mathbb{C}v_1$ if there exists a homogeneous polynomial P_{12} in X_i^+ such that $v_1 = P_{12}\ v_2$. Clearly, in this case we have: $I^1 \subset I^2$, where I^k is the submodule generated by v_k.

The Verma module V^Λ contains a unique proper maximal submodule I^Λ ($\supseteq \tilde{I}^\Lambda$) [11, 13]. Among the lowest-weight modules with lowest weight Λ there is a unique irreducible one, denoted by L_Λ, i.e. $L_\Lambda = V^\Lambda/I^\Lambda$. (If V^Λ is irreducible then $L_\Lambda = V^\Lambda$.)

It may happen that the maximal submodule I^Λ coincides with the submodule $\tilde{I}^\Lambda$ generated by all singular vectors. This is the case, e.g., for all Verma modules if rank $\mathcal{G} \leqslant 2$, or when (5) is fulfilled for all simple roots (and, as a consequence for all positive roots). Here we are interested in the cases when $\tilde{I}^\Lambda$ is a proper submodule of I^Λ. We need the following notion.

Definition. Let V^Λ be a reducible Verma module. A vector $v_{\rm su} \in V^\Lambda$ is called a *subsingular vector* if $v_{\rm su} \notin \tilde{I}^\Lambda$ and the following holds:

$$X\ v_{\rm su} \in \tilde{I}^\Lambda \qquad \forall\ X \in \mathcal{G}^-. \tag{7}$$

Remark 2. The image of a subsingular vector in the factor-module F^Λ is a singular vector of F^Λ. For brevity we shall say the subsingular vector 'becomes' a singular vector in the corresponding factor-module. From this it is also clear that a subsingular vector may be represented by a homogeneous polynomial in X_i^+.

We need to be more explicit even in the general case. First of all it is clear that it is enough for a vector to be subsingular if (7) holds for the negative simple root vectors X_i^-. We can rewrite (7) in the following way:

$$X_i^- \, v_{\rm su} = \sum_{\alpha\in\Delta_i} Q_{i\alpha} \, v^{\alpha,m_\alpha} \tag{8}$$

where $Q_{i\alpha}$ are homogeneous polynomials such that the RHS is a homogeneous polynomial, and Δ_i is a subset of $\Delta_\Lambda \subset \Delta^+$, such that $\alpha \in \Delta_i$ iff $Q_{i\alpha}$ is a non-zero polynomial. Let us denote by $\Delta_{\rm su}$ the union of Δ_i : $\Delta_{\rm su} \equiv \cup_{i=1}^r \Delta^i$. We shall call $\Delta_{\rm su}$ the set of roots *associated* with the subsingular vector $v_{\rm su}$. The corresponding set of singular vectors $\{v^{\alpha,m_\alpha} \mid \alpha \in \Delta_{\rm su}\}$ will be called singular vectors *associated* with the subsingular vector $v_{\rm su}$. Clearly $\Delta_{\rm su}$ is a subset of Δ_Λ and in general a proper subset. Let $I_{\rm su} \equiv \cup_{\alpha\in\Delta_{\rm su}} I^\alpha (\subseteq \tilde I^\Lambda)$, $F_{\rm su} \equiv V^\Lambda / I_{\rm su}$; then $v_{\rm su}$ becomes a singular vector in $F_{\rm su}$, i.e. when we factorize all singular vectors associated with it.

Clearly, if two singular vectors v_1 and v_2 belong to Δ_Λ (Δ_i, $\Delta_{\rm su}$) and v_1 is a descendent of v_2, then we can omit v_1 from the set Δ_Λ (Δ_i, $\Delta_{\rm su}$).

Clearly, $v_{\rm su}$ and $\tilde I^\Lambda$ generate a submodule $I^\Lambda_{\rm su}$ such that

$$I_{\rm su} \subseteq \tilde I^\Lambda \subset I^\Lambda_{\rm su} \subseteq I^\Lambda \subset V^\Lambda . \tag{9}$$

2.2. We now restrict ourselves to $\mathcal{G} = sl(4)$. A synopsis on $U_q(sl(4))$ is given in appendix A. For the six positive roots of the root system of $sl(4)$ one has from (2), (4) (see [7]):

$$m_1 = 1 - \Lambda(H_1) \tag{10a}$$

$$m_2 = 1 - \Lambda(H_2) \tag{10b}$$

$$m_3 = 1 - \Lambda(H_3) \tag{10c}$$

$$m_{12} = 2 - \Lambda(H_{12}) = m_1 + m_2 \tag{10d}$$

$$m_{23} = 2 - \Lambda(H_{23}) = m_2 + m_3 \tag{10e}$$

$$m_{13} = 3 - \Lambda(H_{13}) = m_1 + m_2 + m_3 . \tag{10f}$$

Thus the signature here is $\chi = (m_1, m_2, m_3)$.

For further reference we give the value of the $sl(4)$ second-order Casimir operator [14] in terms of the above notation:

$$C_2 = \tfrac{1}{2}\left(m_{13}^2 + m_2^2 + \tfrac{1}{2}(m_1 - m_3)^2\right) - 5 \tag{11}$$

which is normalized to take zero value on the trivial irrep ($m_k = 1$) (and thus on all representations partially equivalent to it).

2.3. Here we treat the Bernstein–Gel'fand–Gel'fand example of a subsingular vector which appeared in the seminal paper [11] ($q = 1$) and which we give for general q (see appendix A for the relevant $U_q(sl(4))$ formulae). It occurs for $\Lambda(H_1) = \Lambda(H_3) = 1$, $\Lambda(H_2) = 0$, i.e. $\chi = (m_1, m_2, m_3) = (0, 1, 0)$. Thus there are four positive $m_\alpha \in \mathbb{N}$ from (10) : $m_2 = m_{12} = m_{23} = m_{13} = 1$. Correspondingly, there are four singular vectors:

$$v_2 = X_2^+ \, v_0 \qquad m_2 = 1 \tag{12a}$$

$$\begin{aligned} v'_{12} &= X_1^+ \, X_2^+ \, v_0 = X_1^+ \, v_2 \qquad m_{12} = 1 \\ v'_{23} &= X_3^+ \, X_2^+ \, v_0 = X_3^+ \, v_2 \qquad m_{23} = 1 \\ v'_{13} &= X_1^+ \, X_3^+ \, X_2^+ \, v_0 = X_1^+ \, X_3^+ \, v_2 \qquad m_{13} = 1\,. \end{aligned} \tag{12b}$$

However, only the singular vector v_2 is relevant since the others are its descendents.

What is important for us is that there is the following subsingular vector:

$$v_{\rm bgg} = \left(X_1^+ X_2^+ X_3^+ - X_3^+ X_2^+ X_1^+\right) \, v_0\,. \tag{13a}$$

It is easy to see that

$$\begin{aligned} X_1^- \, v_{\rm bgg} &= v'_{23} = X_3^+ \, v_2 \\ X_2^- \, v_{\rm bgg} &= 0 \\ X_3^- \, v_{\rm bgg} &= -v'_{12} = -X_1^+ \, v_2\,. \end{aligned} \tag{14}$$

Thus equation (7) is indeed fulfilled, while comparing with (8) we see that v_2 is indeed associated with $v_{\rm bgg}$.

It is useful for future applications to have a different expression for the subsingular vectors. Equation (13*a*) is in the unordered Chevalley basis. An expression in the ordered PBW basis is:

$$v_{\rm bgg} = \left(X_{13}^+ + q X_3^+ X_{12}^+ + q^{-1} X_{23}^+ X_1^+\right) \, v_0 \tag{13b}$$

which for $q = 1$ is exactly equal to (13*a*) and for $q \neq 1$ differs from (13*a*) by the inessential term $(q - q^{-1}) X_1^+ X_3^+ X_2^+ v_0 \in \tilde{I}^\Lambda$. For $q = 1$ a third expression coinciding with (13*a, b*) is

$$v_{\rm bgg} = \left(X_{12}^+ X_3^+ + X_{23}^+ X_1^+\right) \, v_0 \qquad q = 1\,. \tag{13c}$$

Note that we have translated the result of [11] into our lowest-weight module setting and that the actual expression for $v_{\rm bgg}$ in [11] (given naturally for $q = 1$), is not correct. (Also equations (12*b*) are not given in [11].)

Let $|\widetilde{2}\rangle$ denote the lowest-weight vector of the factor-module $F_2 = V^\Lambda / I_2$. Then the singular vectors in (12) become null conditions, the relevant one (12*a*) giving

$$X_2^+ \, |\widetilde{2}\rangle = 0\,. \tag{15}$$

Clearly, $v_{\rm bgg}$ becomes a singular vector in F_2. If we also factor out $v_{\rm bgg}$ we have the following null conditions in the resulting irreducible module L_Λ with lowest-weight vector $|2\rangle$:

$$X_2^+ \, |2\rangle = 0 \tag{16a}$$

$$\left(X_1^+ X_2^+ X_3^+ - X_3^+ X_2^+ X_1^+\right) \, |2\rangle = 0\,. \tag{16b}$$

2.4. In this and the remaining subsections of section 2 we consider the other archetypal $sl(4)$ example [15, 16]. In this subsection we give some preliminaries. We take first an arbitrary Verma module V^Λ and the following vector:

$$v_f = \mathcal{P}\ v_0 . \tag{17}$$

where $\mathcal{P}$ is the following homogeneous polynomial in $U_q(\mathcal{G}^+)$:

$$\mathcal{P} = X_{13}^+ X_2^+ - q^{-1} X_{12}^+ X_{23}^+ . \tag{18}$$

Below we shall need the following technical result:

$$\begin{aligned} X_2^-\ v_f &= q^{-1}\ \left([\Lambda(H_2)+1]_q X_2^+ X_1^+ - [\Lambda(H_2)]_q X_1^+ X_2^+\right)\ X_3^+\ v_0 \\ &\quad + q^{-1}\ X_3^+\ \left([\Lambda(H_2)-1]_q X_1^+ X_2^+ - [\Lambda(H_2)]_q X_2^+ X_1^+\right)\ v_0 \qquad (19a) \\ &= q^{-1}\ \left([\Lambda(H_2)+1]_q X_2^+ X_3^+ - [\Lambda(H_2)]_q X_3^+ X_2^+\right)\ X_1^+\ v_0 \\ &\quad + q^{-1}\ X_1^+\ \left(-[\Lambda(H_2)-1]_q X_1^+ X_2^+ + [\Lambda(H_2)]_q X_2^+ X_1^+\right)\ v_0 \qquad (19b) \end{aligned}$$

where $[x]_q \equiv \left(q^x - q^{-x}\right)/\lambda$, $\lambda \equiv q - q^{-1}$ (see appendix A).

Also for future reference we note several equivalent forms of the polynomial $\mathcal{P}$ valid for any weight:

$$\begin{aligned} \mathcal{P} &= X_{13}^+ X_2^+ - q^{-1} X_{12}^+ X_{23}^+ \qquad (20a) \\ &= X_{13}^+ X_2^+ - q X_{23}^+ X_{12}^+ \qquad (20b) \\ &= q^{-1}\left(X_1^+ X_2^+ X_3^+ X_2^+ + X_2^+ X_3^+ X_2^+ X_1^+ - [2]_q X_2^+ X_1^+ X_3^+ X_2^+\right) \qquad (20c) \\ &= q^{-1}\left(X_3^+ X_2^+ X_1^+ X_2^+ + X_2^+ X_1^+ X_2^+ X_3^+ - [2]_q X_2^+ X_1^+ X_3^+ X_2^+\right) . \qquad (20d) \end{aligned}$$

and two forms valid if $a \equiv \Lambda(H_2) \neq 1$:

$$\begin{aligned} \mathcal{P} &= \frac{q^{-1}}{[a-1]_q}\ \left(X_3^+ X_2^+ - [2]_q X_2^+ X_3^+\right)\ \left([a-1]_q X_1^+ X_2^+ - [a]_q X_2^+ X_1^+\right) \\ &\quad + \frac{1}{[a-1]_q}\ X_2^+\ \left([a-1]_q X_1^+ X_2^+ - [a]_q X_2^+ X_1^+\right)\ X_3^+ \qquad (21a) \\ &= \frac{q^{-1}}{[a-1]_q}\ \left(X_1^+ X_2^+ - [2]_q X_2^+ X_1^+\right)\ \left([a-1]_q X_3^+ X_2^+ - [a]_q X_2^+ X_3^+\right) \\ &\quad + \frac{1}{[a-1]_q}\ X_2^+\ \left([a-1]_q X_3^+ X_2^+ - [a]_q X_2^+ X_3^+\right)\ X_1^+ . \qquad (21b) \end{aligned}$$

The need for the introduction of the parameter a will become clear below.

2.5. Now consider a Verma module V^Λ with lowest weight Λ satisfying the conditions

$$\Lambda(H_3) = 0 \Longleftrightarrow m_3 = 1 \tag{22a}$$

$$\Lambda(H_1 + H_2) = 1 \Longleftrightarrow m_{12} = 1 . \tag{22b}$$

We shall denote its signature as

$$\chi_1(a) = \chi(\Lambda) = (a, 1-a, 1) , \qquad a = \Lambda(H_2) \in \mathbb{C} \tag{22c}$$

(see equations (10*c*, *d*)). We would like to study this family of representations (and a conjugate one) since for these the (q-) d'Alembert operator will be a (conditionally) invariant operator. This will become clear in sections 3 and 4 while here we find the necessary singular and subsingular vectors.

From the above two conditions follow that there are two singular vectors which are explicitly given by [7, 10]

$$v_3 = X_3^+ \; v_0 \qquad m_3 = 1 \tag{23a}$$

$$v_{12} = \left([a-1]_q X_1^+ X_2^+ - [a]_q X_2^+ X_1^+\right) \; v_0 \qquad m_{12} = 1 . \tag{23b}$$

(There is also a singular vector corresponding to $m_{13} = 2$ [7, 10], which, however, is a descendent of v_3.)

In the above setting we shall show the special place of the vector v_f (which will give rise to the (q-) d'Alembert operator as we shall see in the next sections). We have the following result.

If $a \neq 1$ the vector v_f is a linear combination of descendents of the singular vectors v_3 and v_{12}, while if $a = 1$ the vector v_f is a subsingular vector.

It is straightforward to demonstrate the validity of this statement. Let first $a \neq 1$. Then using (21*a*) we have

$$\begin{aligned} v_f = \mathcal{P} \; v_0 = & \frac{q^{-1}}{[a-1]_q} \; \left(X_3^+ X_2^+ - [2]_q X_2^+ X_3^+\right) \; v_{12} \\ & + \frac{1}{[a-1]_q} X_2^+ \; \left([a-1]_q X_1^+ X_2^+ - [a]_q X_2^+ X_1^+\right) \; v_3 . \end{aligned} \tag{24}$$

To show that for $a = 1$ v_f is a subsingular vector one may use a calculation valid for any a (also using (19*a*)):

$$\begin{aligned} & X_1^- \; v_f = 0 \\ & X_2^- \; v_f = q^{-1} \; \left([a+1]_q X_2^+ X_1^+ - [a]_q X_1^+ X_2^+\right) \; v_3 + q^{-1} \; X_3^+ \; v_{12} \\ & X_3^- \; v_f = 0 \end{aligned} \tag{25}$$

though this calculation obscures the fact that for $a = 1$ the singular vector v_{12} is a descendent one, as we shall see below, where we also show that v_f is not an element of $\tilde{I}^\Lambda$.

We now write down all situations systematically.

2.5.1. If $a \notin \mathbb{Z}$ there are no other nondescendent singular vectors besides (23) and the maximal invariant submodule is $I^\Lambda = I'_1 = I^{\alpha_3} \cup I^{\alpha_{12}}$. We denote by $L'_1 = V^\Lambda / I'_1$ the corresponding irreducible factor-module, and by $|1'\rangle$ the lowest-weight vector of L'_1. Then the expressions in (23) become null conditions, namely we have

$$X_3^+ \; |1'\rangle = 0 \tag{26a}$$

$$\left([a-1]_q X_1^+ X_2^+ - [a]_q X_2^+ X_1^+\right) \; |1'\rangle = 0 . \tag{26b}$$

2.5.2. If $a \in -\mathbb{N}$ then in addition to (23) there is one more singular vector [7, 10] corresponding to $m_2 = 1 - a \in \mathbb{N} + 1$:

$$v_2 = \left(X_2^+\right)^{1-a} \; v_0 \tag{27}$$

and two descendents corresponding to $m_{23} = 2 - a$, $m_{13} = 2$. Thus the maximal invariant submodule is $I^\Lambda = I_1'' = I^{\alpha_3} \cup I^{\alpha_{12}} \cup I^{\alpha_2}$, $L_1'' = V^\Lambda / I_1''$ is the irreducible factor-module, $|1''\rangle$ is the lowest-weight vector of L_1''. Then the null conditions are

$$X_3^+ \; |1''\rangle = 0 \tag{28a}$$

$$\left([a-1]_q X_1^+ X_2^+ - [a]_q X_2^+ X_1^+\right) \; |1''\rangle = 0 \tag{28b}$$

$$\left(X_2^+\right)^{1-a} \; |1''\rangle = 0 . \tag{28c}$$

2.5.3. If $a = 0$ then there is a singular vector corresponding to $m_2 = 1$ and given by (27) with $a = 0$. Here equation (23*b*) is also a descendent and the maximal invariant submodule is generated by the singular vectors (23*a*) and (27), $I^\Lambda = I_1''' = I^{\alpha_3} \cup I^{\alpha_2}$. We denote by $L_1''' = V^\Lambda / I_1'''$ the irreducible factor-module; $|1'''\rangle$ the lowest-weight vector of L_1'''. Then the null conditions are

$$X_3^+ \; |1'''\rangle = 0 \tag{29a}$$

$$X_2^+ \; |1'''\rangle = 0 . \tag{29b}$$

2.5.4. If $a \in \mathbb{N} + 1$ then there exists another singular vector [7, 10]

$$v_1 = \left(X_1^+\right)^a \; v_0 \tag{30}$$

Thus the maximal invariant submodule is $I^\Lambda = I_1^{IV} = I^{\alpha_3} \cup I^{\alpha_{12}} \cup I^{\alpha_1}$, $L_1^{IV} = V^\Lambda / I_1^{IV}$ is the irreducible factor-module, $|1^{IV}\rangle$ is the lowest-weight vector of L_1^{IV}. Then the null conditions are

$$X_3^+ \; |1^{IV}\rangle = 0 \tag{31a}$$

$$\left([a-1]_q X_1^+ X_2^+ - [a]_q X_2^+ X_1^+\right) \; |1^{IV}\rangle = 0 \tag{31b}$$

$$\left(X_1^+\right)^a \; |1^{IV}\rangle = 0 . \tag{31c}$$

2.5.5. Finally, if $a = 1$ then the non-descendent singular vectors are $v_3 = X_3^+ v_0$, see (23*a*), and $v_1 = X_1^+ v_0$, see (30) with $a = 1$, while (23*b*) is a descendent of (30), and there also appears a singular vector v_{23}', see (12*b*), corresponding to $m_{23} = 1$ which is a descendent to (23*a*). Here we also have the subsingular vector v_f, see (17), (25), from the latter the essential one simplifying here to

$$X_2^- \; v_f = q^{-1} \; \left([2]_q X_2^+ X_1^+ - X_1^+ X_2^+\right) \; v_3 - q^{-1} \; X_3^+ \; X_2^+ \; v_1 . \tag{32}$$

Now it remains from the above proof to show that v_f cannot be represented as a linear combination of descendents of v_1 and v_3, and thus does not belong to $\tilde{I}^\Lambda$, which is also easy to see by inspecting (20).

We denote by $\tilde{I}^\Lambda = I^{\alpha_1} \cup I^{\alpha_3}$ the submodule generated by these singular vectors, by $F_1 = V^\Lambda/\tilde{I}^\Lambda$ the factor-module, by $|\widetilde{1}\rangle$ the lowest-weight vector of F_1. We have the following null conditions in F_1:

$$X_3^+ \, |\widetilde{1}\rangle = 0 \tag{33a}$$

$$X_1^+ \, |\widetilde{1}\rangle = 0 \,. \tag{33b}$$

The vector v_f becomes a singular vector in F_1 which we denote as

$$v_{f_1} = \mathcal{P} \, |\widetilde{1}\rangle \,. \tag{34}$$

Factoring out the submodule built on v_{f_1} we obtain the the irreducible factor-module $L_1 = V^\Lambda/I_1^\Lambda$. We denote by $|1\rangle$ the lowest-weight vector of L_1. Then the null conditions are

$$X_3^+ \, |1\rangle = 0 \tag{35a}$$

$$X_1^+ \, |1\rangle = 0 \tag{35b}$$

$$\left(X_3^+X_2^+ - [2]_q X_2^+X_3^+\right) X_1^+X_2^+ \, |1\rangle = 0 \tag{35c}$$

where for (35c) we have used (35a) and (20d). An equivalent condition to (35c) is

$$\left(X_1^+X_2^+ - [2]_q X_2^+X_1^+\right) X_3^+X_2^+ \, |1\rangle = 0 \tag{35c$'$}$$

where we have used (35b) and (20c).

Conditions (31) and (35) (conditions (35c, c') in a different, but equivalent form) were given first in [17]. The corresponding irreps (for $a \in \mathbb{N}$) were shown [17] to be a construction of the irreducible massless representations of a q-conformal algebra (with $|q| = 1$) characterized by the helicity $h = (a-1)/2 \in \frac{1}{2}\mathbb{Z}_+$.

2.6. Analogously consider a Verma module V^Λ with lowest weight Λ satisfying the conditions

$$\Lambda(H_1) = 0 \iff m_1 = 1 \tag{36a}$$

$$\Lambda(H_2 + H_3) = 1 \iff m_{23} = 1 \tag{36b}$$

$$\chi_3(a) = \chi(\Lambda) = (1, 1-a, a) \qquad a = \Lambda(H_2) \in \mathbb{C} \tag{36c}$$

(see equations (10a, d)). This case is conjugate to that considered in subsection 2.5 and all statements and formulae may be obtained *verbatim* by exchanging indices $1 \longleftrightarrow 3$, $12 \longleftrightarrow 23$. Thus, we shall give for future reference only the final formulae analogous to (31). Namely, the conditions fulfilled in the irreducible lowest-weight module L_3 (with $a \in \mathbb{N}+1$) are

$$X_1^+ \, |3\rangle = 0 \tag{37a}$$

$$\left([a-1]_q X_3^+X_2^+ - [a]_q X_2^+X_3^+\right) |3\rangle = 0 \tag{37b}$$

$$\left(X_3^+\right)^a \, |3\rangle = 0 \,. \tag{37c}$$

Conditions (37) were given first in [17].

It is interesting to note that a lowest weight can satisfy both (22) and (36) which happens only for the special case $a = 1$, which was considered in the previous subsection.

3. Conditionally invariant equations

3.1. We now write down explicitly the conditionally invariant equations related to the subsingular vectors considered in the previous section. For simplicity we treat the case $q = 1$ first and the q-deformed analogues in the next section.

We use the approach of [7] which we give in a condensed form here. We work with induced representations, called *elementary representations* (ERs). The functions of the ERs can be taken to be complex-valued C^∞ functions on the group G. The representation action is given by the *left regular action*, which in infinitesimal form is

$$(X_L\varphi)(g) \equiv \frac{\mathrm{d}}{\mathrm{d}t}\varphi(\exp(-tX)g)|_{t=0} \tag{38}$$

where $X \in \mathcal{G}$, $g \in G$, $\mathcal{G}$ is the Lie algebra of G. These functions possess the properties of right covariance [7]. For our purposes it is enough to consider *holomorphic* elementary representations for which right covariance means

$$\hat{X}\varphi = \Lambda(X)\cdot\varphi \qquad X \in \mathcal{H} \tag{39a}$$

$$\hat{X}\varphi = 0 \qquad X \in \mathcal{G}^- \tag{39b}$$

where $\Lambda \in \mathcal{H}^*$, and $\hat{X}$ is the *right* action of the generators of the algebra $\mathcal{G}$:

$$(\hat{X}\varphi)(g) \;\equiv\; \frac{\mathrm{d}}{\mathrm{d}t}\varphi(g\exp(tX))|_{t=0}\,. \tag{40}$$

Right covariance is also used to pass from functions on the group G to the so-called *reduced* functions $\hat{\varphi}$ on the coset space G/B, where $B = \exp(\mathcal{H})\exp(\mathcal{G}^-)$ is a Borel subgroup of G. Note that G/B is a completion of $G^+ = \exp(\mathcal{G}^+)$ and in practical calculations one is usually using the local coordinates of G^+.

The weight Λ completely characterizes these representations, which we denote by C^Λ, each of which is then in correspondence with the lowest-weight representations with the same lowest weight, in particular, with the Verma module V^Λ.

Now the main ingredient of the procedure of [7] is that to every singular vector there corresponds an intertwining differential operator. Namely, to the singular vector $v_s = v^{\alpha,m_\alpha}$ (see equation (6)) of the Verma module V^Λ there corresponds an intertwining differential operator

$$D^{\alpha,m_\alpha} \;:\; C^\Lambda \;\longrightarrow\; C^{\Lambda+m_\alpha\alpha} \tag{41}$$

given explicitly by

$$D^{\alpha,m_\alpha} = \mathcal{P}^{\alpha,m}(\hat{X}_1^+,\ldots,\hat{X}_r^+) \tag{42}$$

where $\mathcal{P}^{\alpha,m}$ is the same polynomial as in (6), and $\hat{X}_i^+$ is the right action (40). This operator gives rise to the $\mathcal{G}$-invariant equation

$$D^{\alpha,m_\alpha}\,\hat{\varphi} = \hat{\varphi}' \qquad \hat{\varphi} \in C^\Lambda \quad \hat{\varphi}' \in C^{\Lambda+m_\alpha\alpha}\,. \tag{43}$$

In the same way a subsingular vector produces a differential operator and equation which are conditionally invariant. The latter means that this invariance hold only on the intersection of the kernels of all intertwining operators D^{α,m_α} such that α and the singular vectors v^{α,m_α} are associated with the singular vector in question, i.e. on the space

$$C_{\mathrm{su}} = \{\hat{\varphi} \in C^\Lambda | D^{\alpha,m_\alpha}\,\hat{\varphi} = 0\;,\;\forall\,\alpha \in \Delta_{\mathrm{su}}\} \tag{44}$$

(see subsection 2.1.). A conditionally invariant equation has non-trivial RHS if we take the situation corresponding to the reducible factor-module $F^\Lambda = V^\Lambda/\tilde{I}^\Lambda$; the latter is realized when we do not impose in F^Λ the null condition corresponding to the subsingular vector which in F^Λ is a singular vector. A conditionally invariant equation has trivial RHS if we take the situation corresponding to the irreducible factor-module $L_\Lambda = V^\Lambda/I^\Lambda$, i.e. if we impose in F^Λ the null condition corresponding to the subsingular vector. Below we consider both situations, for which we are prepared by the detailed analysis of section 2.

Remark 3. Note that one may exchange the left and right actions in the above considerations, i.e. consider the representations acting as *right* regular representations with properties of *left* covariance. Independently, if one uses *highest*-weight representations (see remark 1) one then uses the coset G/B', where $B' = \exp(\mathcal{H})\exp(\mathcal{G}^+)$ is the Borel subgroup of G conjugate to B.

Remark 4. As we noted if one wants to treat the case of a real non-compact algebra $\mathcal{G}_0$ one also has to use the results for its complexification $\mathcal{G}$. The application of these results to $\mathcal{G}_0$ has some subtleties [7]. However, in the case at hand when $\mathcal{G}_0 = su(2,2)$ and $\mathcal{G} = sl(4)$ the passage to $su(2,2)$ is straightforward [18]. Also considering representations of the corresponding groups (which are used here only to provide the representation spaces) involves some subtleties [7], which, however, are not felt in the case under consideration [18].

Referring further the general case to [7] here we restrict ourselves to $\mathcal{G} = sl(4)$, $G = SL(4)$. We pass to functions on the flag manifold $\mathcal{Y} = SL(4)/B$, where B is the Borel subgroup of $SL(4)$ consisting of all upper diagonal matrices. (Equally well one may take the flag manifold $SL(4)/B'$, where B' is the Borel subgroup of lower diagonal matrices.) We denote the six local coordinates on $\mathcal{Y}$ by $x_\pm, v, \bar{v}, z, \bar{z}$. From the explicit form of the singular vectors it is clear that we need only the right action of the three simple root generators. Denoting this right action of X_k^+ by R_k, from [7] we have

$$\begin{aligned} R_1 &= \partial_z \equiv \frac{\partial}{\partial z} \\ R_2 &= \bar{z}z\partial_+ + z\partial_v + \bar{z}\partial_{\bar{v}} + \partial_- \\ R_3 &= \partial_{\bar{z}} \equiv \frac{\partial}{\partial \bar{z}} \end{aligned} \tag{45}$$

where

$$\partial_\pm \equiv \frac{\partial}{\partial x_\pm} \qquad \partial_v \equiv \frac{\partial}{\partial v} \qquad \partial_{\bar{v}} \equiv \frac{\partial}{\partial \bar{v}}\,. \tag{46}$$

Things are arranged so that in the conformal setting we can use the same coordinates [18]. In this case the coordinates $x_\pm, v, \bar{v}$ are related to the Minkowski space–time coordinates x_0, x_1, x_2, x_3:

$$x_\pm \equiv x_0 \pm x_3 \qquad v \equiv x_1 - \mathrm{i}x_2 \qquad \bar{v} \equiv x_1 + \mathrm{i}x_2 \tag{47}$$

while $z, \bar{z}$ encode the inducing Lorentz representation as explained below. In particular, one may use the following covariant representation for R_2 [18] employing the Pauli matrices σ_μ:

$$R_2 = (\bar{z} \quad 1)\ \sigma_\mu \partial^\mu \begin{pmatrix} z \\ 1 \end{pmatrix}$$
$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{48}$$
$$\sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note also that under the natural conjugation

$$\omega(x_\pm) = x_\pm \qquad \omega(v) = \bar{v} \qquad \omega(z) = \bar{z} \tag{49}$$

$\mathcal{Y}$ is also a flag manifold of the conformal group $SU(2,2)$.

The reduced function spaces of the ERs in which our equations are defined are complex-valued C^∞ functions on the flag manifold. The holomorphic ERs of $sl(4)$ are labelled by the signature $\chi = (m_1, m_2, m_3)$. We give the explicit expressions of the representation action for $U_q(sl(4))$ in appendix B, from which those for $sl(4)$ may be obtained by setting $q = 1$.

In the $su(2,2)$ case most applications in physics are in the case when $m_1, m_3 \in \mathbb{N}$ and one uses reduced functions which are polynomials in the variables $z, \bar{z}$ of degrees $m_1 - 1, m_3 - 1$, respectively. These then carry finite-dimensional irreducible representations of the Lorentz algebra of dimension $m_1 m_3$. Let us stress that this is an indexless notation on which all Lorentz components of the fields are gathered together by the polynomial dependence in $z, \bar{z}$. To restore the components one has to take the entries of the independent terms in $z, \bar{z}$, see [18]. Note that in the physics literature, instead of (m_1, m_2, m_3), the labelling $[d, j_1, j_2]$ is often used, where $d = 2 - (m_{13} + m_2)/2$ is the conformal weight, $j_1 = (m_1 - 1)/2$, $j_2 = (m_3 - 1)/2$, so that for finite-dimensional Lorentz irreps one has $j_k \in \mathbb{Z}_+/2$.

3.2. We start with the equations arising from the BGG example of a subsingular vector. Substituting (45) in (15) we obtain the following $sl(4)$- and $su(2,2)$-invariant equation:

$$R_2\ \hat{\varphi} = \left(\bar{z} z \partial_+ + z \partial_v + \bar{z} \partial_{\bar{v}} + \partial_-\right) \hat{\varphi} = 0 \tag{50a}$$

while the subsingular vector v_{bgg} gives rise to the following conditionally invariant equation:

$$\left(R_1 R_2 R_3 - R_3 R_2 R_1\right) \hat{\varphi} = \left(\partial_v \partial_{\bar{z}} - \partial_{\bar{v}} \partial_z + \left(\bar{z} \partial_{\bar{z}} - z \partial_z\right) \partial_+\right) \hat{\varphi} = \hat{\varphi}' \tag{50b}$$

where $\hat{\varphi} \in C^\Lambda$ and satisfies (50*a*), $\hat{\varphi}' \in C^{\Lambda'}$, $\Lambda' = \Lambda - \alpha_{13}$, the corresponding signatures being $\chi = (0, 1, 0)$, $\chi' = (-1, 1, -1)$. (Note that the second Casimir operator has the same value in the two representations: $C_2(\chi) = C_2(\chi') = -4$, see equation (11).) If we consider the irreducible factor-module L_Λ, which means that we should use (16) instead of (15), instead of (50*b*) we have

$$\left(\partial_v \partial_{\bar{z}} - \partial_{\bar{v}} \partial_z + \left(\bar{z} \partial_{\bar{z}} - z \partial_z\right) \partial_+\right) \hat{\varphi} = 0. \tag{50c}$$

3.3. We now turn to equations arising from the other archetypal $sl(4)$ example. We consider the case when the lowest weight satisfies conditions (22).

We shall substitute the operators R_k in the null conditions (26), (28), (29), (31), (33), (35). In all cases arising from the singular vector $v_3 = X_3^+ v_0$, (null conditions (26*a*), (28*a*), (29*a*), (31*a*), (33*a*), (35*a*)) we have the equation

$$R_3 \, \hat\varphi = \partial_{\bar z} \, \hat\varphi = 0 \tag{51}$$

which means that our functions do not depend on the variable $\bar z$—this is valid for the signature $\chi_1(a)$ and arbitrary a. (In the conjugate situation with signature $\chi_3(a)$ our functions do not depend on the variable z.)

Furthermore, we have the equations arising from the singular vector v_2, when $a \in \mathbb{Z}_-$ (null conditions (28*c*), (29*b*)):

$$(R_2)^{1-a} \, \hat\varphi = 0 \qquad a \in \mathbb{Z}_- . \tag{52}$$

Next, we have the equations arising from the singular vector v_1, when $a \in \mathbb{N}$ (null conditions (31*c*), (33*b*), (35*b*)):

$$(\partial_z)^a \, \hat\varphi = 0 \qquad a \in \mathbb{N} \tag{53}$$

which means that our functions are polynomials in the variable z of degree $a - 1$. Thus for $a = 1$ our functions also do not depend on z.

Next we write down the equation arising from the singular vector v_{12} (null conditions (26*b*), (28*b*), (31*b*)):

$$\Big((a-1)R_1R_2 - aR_2R_1\Big) \, \hat\varphi = \Big((a-1)(\partial_v + \bar z\partial_+) - R_2\partial_z\Big) \, \hat\varphi = 0 . \tag{54}$$

It is also valid in all cases: however, for $a = 0$ it follows from (52) and for $a = 1$ it follows from (53). Now, since (54) is a first degree polynomial in $\bar z$, on which our functions do not depend, it actually consists of two equations, though not invariant by themselves, i.e. we have

$$\Big((a-1-z\partial_z)\partial_v - \partial_-\partial_z\Big) \, \hat\varphi = 0 \tag{55a}$$

$$\Big((a-1-z\partial_z)\partial_+ - \partial_{\bar v}\partial_z\Big) \, \hat\varphi = 0 . \tag{55b}$$

Finally, we obtain the conditionally invariant equations corresponding to the subsingular vector v_f. Let us denote by $\hat{\mathcal{P}}$ the polynomial $\mathcal{P}$ with X_k^+ replaced by R_k. Now we shall obtain this operator in explicit form:

$$\hat{\mathcal{P}} \, \hat\varphi = (R_3R_2 - 2R_2R_3) \;\; R_1 \;\; R_2 \, \hat\varphi \tag{56a}$$

$$= (z\partial_+ + \partial_{\bar v} - R_2\partial_{\bar z}) \;\; \partial_z \;\; R_2 \, \hat\varphi \tag{56b}$$

$$= ((z\partial_+ + \partial_{\bar v}) \; \partial_z \; R_2 - R_2 \; \partial_z \; (z\partial_+ + \partial_{\bar v})) \; \hat\varphi \tag{56c}$$

$$= (\partial_{\bar v} \; \partial_v - \partial_-\partial_+) \;\; \hat\varphi = \Box \, \hat\varphi \tag{56d}$$

where we used (51) in passing from (56*b*) to (56*c*). Thus, we have recovered the d'Alembert operator. Note that (56) is valid for arbitrary a since we have used only condition (51) which is valid for all of our representations.

Now if for $a = 1$ we take only invariant equations arising from the conditions (33) (i.e. we work with the counterpart of the factor-module F_1), we have the following system of differential equations:

$$\partial_{\bar z} \, \hat\varphi = 0 \tag{57a}$$

$$\partial_z \, \hat\varphi = 0 \tag{57b}$$

$$\Box \, \hat\varphi = \hat\varphi' \tag{57c}$$

where $\hat{\varphi} \in C^{\Lambda}$ and satisfies (57*a*, *b*), $\hat{\varphi}' \in C^{\Lambda'}$, $\Lambda' = \Lambda - \alpha_{13} - \alpha_2$, the corresponding signatures being $\chi = (1,0,1)$, $\chi' = (1,-2,1)$. (Note that the second Casimir operator has the same value in the two representations: $C_2(\chi) = C_2(\chi') = -3$, see (11).) If we consider the irreducible factor-module L_1, which means that we should use (35) instead of (33), instead of (57*c*) we have

$$\Box\, \hat{\varphi} = 0 \tag{57d}$$

where $\hat{\varphi}$ is as in (57*c*) and again satisfies (57*a*, *b*).

Thus, from the subsingular vector v_f we have obtained the d'Alembert equations (57c, d) as conditionally sl(4) and su(2,2) invariant equations.

Now we turn to the cases when $a \neq 1$. In these cases the vector v_f is a linear combination of the singular vectors v_1 and v_{12} and it becomes zero when these singular vectors are factorized. Since v_f gives rise to the d'Alembert operators for all a we expect that the d'Alembert equation (57*d*) will hold automatically if the invariant equations (51), (54) (arising from v_1, v_{12}) hold. This is indeed so. We use the two equations (55) which are the two components of (54). First we take $\partial_{\bar{v}}$ derivative from (55*a*) and ∂_- derivative from (55*b*) and subtracting the two we get

$$(a-1-z\partial_z)\left(\partial_-\partial_+ - \partial_{\bar{v}}\partial_v\right)\hat{\varphi} = (a-1-z\partial_z)\,\Box\,\hat{\varphi} = 0\,. \tag{58a}$$

This still follows from (57*d*). Analogously, taking ∂_+ derivative from (55*a*) and ∂_v derivative from (55*b*) and subtracting the two we get

$$\partial_z\left(\partial_-\partial_+ - \partial_{\bar{v}}\partial_v\right)\hat{\varphi} = \partial_z\,\Box\,\hat{\varphi} = 0\,. \tag{58b}$$

This also follows from (57*d*). Now, clearly from (58*a*, *b*) it follows that:

$$(a-1)\,\Box\,\hat{\varphi} = 0 \tag{58c}$$

which implies the d'Alembert equation if $a \neq 1$.

Using the conjugate situation with signature $\chi_3(a)$ we recover the d'Alembert equation on functions which do not depend on z and satisfy

$$\Big((a-1)R_3R_2 - aR_3R_1\Big)\,\hat{\varphi} = \Big((a-1)(\partial_{\bar{v}} + z\partial_+) - R_2\partial_{\bar{z}}\Big)\,\hat{\varphi} = 0 \tag{59}$$

instead of (54). Furthermore the analogues of (55*a*, *b*), (51), (53), respectively, are

$$\Big((a-1-\bar{z}\partial_{\bar{z}})\partial_{\bar{v}} - \partial_-\partial_{\bar{z}}\Big)\,\hat{\varphi} = 0 \tag{60a}$$

$$\Big((a-1-\bar{z}\partial_{\bar{z}})\partial_+ - \partial_v\partial_{\bar{z}}\Big)\,\hat{\varphi} = 0 \tag{60b}$$

$$\partial_z\,\hat{\varphi} = 0 \tag{60c}$$

$$(\partial_{\bar{z}})^a\,\hat{\varphi} = 0 \qquad a \in \mathbb{N}\,. \tag{60d}$$

Thus if $a \in \mathbb{N}$, then the functions of the irreducible representations are polynomials in $\bar{z}$ of degree $a-1$.

If $a \in \mathbb{Z}_-$ our functions satisfy (52) as those with signature $\chi_1(a)$.

Finally the d'Alembert equation (57*d*) follows from equations (60*a*, *b*) ($a \neq 1$). We do not need to consider $a = 1$ since the two signatures coincide.

We now summarize the results of this subsection. The first result is that the d'Alembert equation (1) (equation (57*d*)) holds in the representation spaces with signatures $\chi_1(a) = (a, 1-a, 1)$ ($\chi_3(a) = (1, 1-a, a)$) if our functions do not depend on the variable $\bar{z}$ (z) and in addition satisfy equations (57*a*, *b*) (equations (60*a*, *b*)). For $a = 1$ the d'Alembert equations (57*c*, *d*) are conditionally $sl(4)$ and $su(2,2)$ invariant, while for $a \neq 1$ the d'Alembert equation (57*d*) just follows from equations (57*a*, *b*) (equations (60*a*, *b*)). If $a \in \mathbb{N}$ then our functions are polynomials in z ($\bar{z}$), of degree $a - 1$.

In the $su(2,2)$ setting we again recall that the variables $z, \bar{z}$ are representing the *spin dependence coming from the Lorentz representation [18–20]. The above result is* then restated thus in the case $a \notin \mathbb{N}$: the d'Alembert equation holds if the fields carry holomorphic (depending only on z) or antiholomorphic (depending only on $\bar{z}$) infinite-dimensional representations of the Lorentz algebra; in addition they satisfy (57*a*, *b*) and (60*a*, *b*), respectively. In the case $a \in \mathbb{N}$ we restrict ourselves to Lorentz representations which are finite-dimensional; in fact, of dimension a.

The case $a = 1$ is remarkable in one more respect, namely, in this case one may have a non-trivial RHS, see (57*c*). It is easy to check that there are no other cases with non-trivial RHS. In fact, for $a \neq 1$ (57*d*) follows from (57*a*, *b*), or (60*a*, *b*). This can also be shown independently. Indeed, in the first case the candidate signatures would be: $\chi_1(a) = (a, 1-a, 1)$, $\chi_1'(a) = (a, -1-a, 1)$. We know that a necessary condition to have an invariant equation is that the two representations would have the same Casimir operators, in particular, one should have $C_2(\chi_1(a)) - C_2(\chi_1'(a)) = 0$, where C_2 is given in (11). Calculating this difference we obtain

$$C_2(\chi_1(a)) - C_2(\chi_1'(a)) = 2(a-1) \tag{61}$$

which is not zero unless $a = 1$.

The cases $a > 1$ are interesting in other contexts, especially, if we consider together the representations with the conjugated signatures $\chi_1(a)$ and $\chi_3(a)$ with the same $a \in \mathbb{N}+1$. In particular, in the case $a = 2$ the two conjugated fields are two-component spinors and (54), (59) are the two conjugated Weyl equations.

The cases $a = 3$ are maybe the most interesting. The Lorentz dimension is 6 ($= 2a$) and the resulting field is the Maxwell field. As was shown in detail in [19] equations (54), (59) are just a rewriting of the free Maxwell equations

$$\partial^\mu F_{\mu\nu} = 0 \qquad \partial^\mu {}^*F_{\mu\nu} = 0. \tag{62}$$

Remark 5. The general Maxwell equations with non-zero current were considered in [19], i.e.

$$\partial^\mu F_{\mu\nu} = J_\nu \qquad \partial^\mu {}^*F_{\mu\nu} = 0 \tag{63}$$

which are then equivalent to a modification of (54), (59) with non-zero right-hand sides *which are given explicitly in equations (5a, b) of [19]. More than that, in [19] is discussed* an hierarchy of Maxwell equations involving two conjugated families of representations: $\chi_n^+ = (n+3, -n-2, n+1)$, $\chi_n^- = (n+1, -n-2, n+3)$, $n \in \mathbb{Z}_+$, from which the Maxwell case is obtained for $n = 0$. Note that there is no other intersection of this Maxwell hierarchy with the two families $\chi_1(a)$ and $\chi_3(a)$ (see equations (22*c*), (36*c*)) which we consider in this paper.

We may write out many other equations with indices, however, one of the main points here is that in this form equations (57) and (60) are valid different representation spaces, the different representations manifesting themselves only through the parameter a.

Remark 6. (i) It is interesting to note that there are other conditionally invariant equations involving the d'Alembert operator, from which (57*c*) is a partial case ($m = 1$), namely

$$\Box^m \hat{\varphi} = \hat{\varphi}' \qquad m \in \mathbb{N} \tag{64}$$

where $\hat{\varphi} \in C^{\Lambda}$, $\hat{\varphi}' \in C^{\Lambda'}$, $\Lambda' = \Lambda - m(\alpha_{13} + \alpha_2)$, the corresponding signatures being $\chi = (m, 0, m)$, $\chi' = (m, -2m, m)$. These are produced by subsingular vectors of weights $m(\alpha_{13} + \alpha_2)$ [16]. The functions $\hat{\varphi}$, $\hat{\varphi}'$ carry irreducible Lorentz representations which are symmetric traceless tensors of rank $m - 1$. (For early examples, namely, (64) with $m = 2$, obtained from other considerations, see [3–5].)

(ii) We should note that there are conditionally invariant equations involving the d'Alembert operator, which do not arise from subsingular vectors but from reduction of *integral* intertwining operators. These equations are also given by (64), however, the corresponding signatures are $\chi = (m, n, m)$, $\chi' = (m, n - 2m, m)$, $m, n \in \mathbb{N}$, see [6], for example.

(iii) We should note that in (most of) the physical applications equation (64) is not considered conditionally invariant. The reason is that only representations induced from finite-dimensional Lorentz representations are considered there. The fact that these representations are also subspaces of reducible representations is ignored and thus the restriction to these subspaces is not considered to be a condition (see [2–6]).

4. Conditionally invariant q-difference equations

4.1. We now give the treatment of the conditionally invariant equations in the q-deformed case. First we need to introduce our reduced representation spaces C^{Λ} with signatures $\chi = \chi(\Lambda) = (m_1, m_2, m_3)$, cf [21, 19]. The elements of C^{Λ}, which we shall call (abusing the notion) functions, are formal power series in the non-commuting variables $z, v, x_-, x_+, \bar{v}, \bar{z}$, which generate the q-deformation $\mathcal{Y}_q$ of the flag manifold $\mathcal{Y}$ (the commutation relations of these variables using the same notation are given in [19]). More explicitly, these reduced functions are given by

$$\begin{aligned} \hat{\varphi}(\bar{Y}) &= \sum_{i,j,k,\ell,m,n \in \mathbb{Z}_+} \mu_{ijk\ell mn}\, \hat{\varphi}_{ijk\ell mn} \\ \hat{\varphi}_{ijk\ell mn} &= z^i\, v^j\, x_-^k\, x_+^\ell\, \bar{v}^m\, \bar{z}^n \end{aligned} \tag{65}$$

where $\bar{Y}$ denotes the set of the six variables.

Next we introduce the following operators acting on our functions:

$$\hat{M}_\kappa\, \hat{\varphi}(\bar{Y}) = \sum_{i,j,k,\ell,m,n \in \mathbb{Z}_+} \mu_{ijk\ell mn}\, \hat{M}_\kappa\, \hat{\varphi}_{ijk\ell mn} \tag{66a}$$

$$T_\kappa\, \hat{\varphi}(\bar{Y}) = \sum_{i,j,k,\ell,m,n \in \mathbb{Z}_+} \mu_{ijk\ell mn}\, T_\kappa\, \hat{\varphi}_{ijk\ell mn} \tag{66b}$$

where $\kappa = z, \pm, v, \bar{v}, \bar{z}$, and the explicit action on $\hat{\varphi}_{ijk\ell mn}$ is defined by

$$\hat{M}_z\, \hat{\varphi}_{ijk\ell mn} = \hat{\varphi}_{i+1,jk\ell mn} \tag{67a}$$

$$\hat{M}_v\, \hat{\varphi}_{ijk\ell mn} = \hat{\varphi}_{i,j+1,k\ell mn} \tag{67b}$$

$$\hat{M}_-\, \hat{\varphi}_{ijk\ell mn} = \hat{\varphi}_{ij,k+1,\ell mn} \tag{67c}$$

$$\hat{M}_+\, \hat{\varphi}_{ijk\ell mn} = \hat{\varphi}_{ijk,\ell+1,mn} \tag{67d}$$

$$\hat{M}_{\bar{v}}\ \hat{\varphi}_{ijk\ell mn} = \hat{\varphi}_{ijk\ell,m+1,n} \tag{67e}$$

$$\hat{M}_{\bar{z}}\ \hat{\varphi}_{ijk\ell mn} = \hat{\varphi}_{ijk\ell m,n+1} \tag{67f}$$

$$T_z\ \hat{\varphi}_{ijk\ell mn} = q^i\ \hat{\varphi}_{ijk\ell mn} \tag{68a}$$

$$T_v\ \hat{\varphi}_{ijk\ell mn} = q^j\ \hat{\varphi}_{ijk\ell mn} \tag{68b}$$

$$T_-\ \hat{\varphi}_{ijk\ell mn} = q^k\ \hat{\varphi}_{ijk\ell mn} \tag{68c}$$

$$T_+\ \hat{\varphi}_{ijk\ell mn} = q^\ell\ \hat{\varphi}_{ijk\ell mn} \tag{68d}$$

$$T_{\bar{v}}\ \hat{\varphi}_{ijk\ell mn} = q^m\ \hat{\varphi}_{ijk\ell mn} \tag{68e}$$

$$T_{\bar{z}}\ \hat{\varphi}_{ijk\ell mn} = q^n\ \hat{\varphi}_{ijk\ell mn}\,. \tag{68f}$$

Now we define the q-difference operators by

$$\hat{\mathcal{D}}_\kappa\ \hat{\varphi}(\bar{Y}) = \frac{1}{\lambda}\ \hat{M}_\kappa^{-1}\ \left(T_\kappa - T_\kappa^{-1}\right)\ \hat{\varphi}(\bar{Y}) \qquad \kappa = z, \pm, v, \bar{v}, \bar{z}\,. \tag{69}$$

Note that although $\hat{M}_\kappa^{-1}$ is not defined if the corresponding variable is of zero degree, the operator $\hat{\mathcal{D}}_\kappa$ is well defined on such terms, and the result is zero (given by the action of $(T_\kappa - T_\kappa^{-1})$). Of course, for $q \to 1$ we have $\hat{\mathcal{D}}_\kappa \to \partial_\kappa$.

Using the above operators the representation (left) action was given in [21] for general n and for $n = 4$ in [20]; for the reader's convenience it is summarized in appendix B.

The q-difference analogues of the operators R_k, i.e. the *right* action of $U_q(sl(4))$ on our functions, are also known from [21]. Adapting this to our notation we have

$$R_1^q = \hat{\mathcal{D}}_z\ T_z\ (T_v T_- T_+ T_{\bar{v}})^{-1} \tag{70a}$$

$$\begin{aligned} R_2^q = \Big(&q\ \hat{M}_z\ \hat{\mathcal{D}}_v\ T_v^2\ T_-^2 + \hat{\mathcal{D}}_-\ T_- + \hat{M}_z\ \hat{M}_{\bar{z}}\ \hat{\mathcal{D}}_+\ (T_v T_-)^{-1}\ T_{\bar{v}} + q^{-1}\ \hat{M}_{\bar{z}}\ \hat{\mathcal{D}}_{\bar{v}} \\ &- \lambda\ \hat{M}_v\ \hat{M}_{\bar{z}}\ \hat{\mathcal{D}}_-\ \hat{\mathcal{D}}_+\ T_{\bar{v}}\Big)\ T_{\bar{v}}\ T_{\bar{z}}^{-1} \end{aligned} \tag{70b}$$

$$R_3^q = \hat{\mathcal{D}}_{\bar{z}}\ T_{\bar{z}}\,. \tag{70c}$$

To obtain the (conditionally) invariant q-difference equations amounts now simply to substituting X_k^+ with R_k^q in the expressions of the (sub)singular vectors for general q.

4.2. Substituting equation (70) in (15) we obtain the following $U_q(sl(4))$ and $U_q(su(2,2))$ invariant equation:

$$\begin{aligned} R_2^q\ \hat{\varphi} = \Big(&q\ \hat{M}_z\ \hat{\mathcal{D}}_v\ T_v^2\ T_-^2 + \hat{\mathcal{D}}_-\ T_- + \hat{M}_z\ \hat{M}_{\bar{z}}\ \hat{\mathcal{D}}_+\ (T_v T_-)^{-1}\ T_{\bar{v}} + q^{-1}\ \hat{M}_{\bar{z}}\ \hat{\mathcal{D}}_{\bar{v}} \\ &- \lambda\ \hat{M}_v\ \hat{M}_{\bar{z}}\ \hat{\mathcal{D}}_-\ \hat{\mathcal{D}}_+\ T_{\bar{v}}\Big)\ T_{\bar{v}}\ T_{\bar{z}}^{-1}\ \hat{\varphi} = 0\,. \end{aligned} \tag{71a}$$

The subsingular vector v_{bgg} gives rise to the following conditionally invariant equation:

$$\begin{aligned} &\left(R_1^q\ R_2^q\ R_3^q - R_3^q\ R_2^q\ R_1^q\right)\ \hat{\varphi} \\ &\quad = \Big\{q^4\ \hat{\mathcal{D}}_v\ \hat{\mathcal{D}}_{\bar{z}}\ T_z^2\ T_v^2\ T_-^2 - \hat{\mathcal{D}}_{\bar{v}}\ \hat{\mathcal{D}}_z\ T_z\ T_{\bar{z}} \end{aligned}$$

$$+q^2\,\hat{\mathcal{D}}_+\,T_+\,T_{\bar v}^2\left(T_z\,T_{\bar z}+\left(q^{-1}\,T_{\bar z}\,T_z^{-1}-q\,T_z\,T_{\bar z}^{-1}\right)/\lambda\right)$$
$$+q\lambda\,\hat{M}_v\,\hat{\mathcal{D}}_-\,\hat{\mathcal{D}}_+\,\hat{\mathcal{D}}_z\,T_z\,T_{\bar z}\,T_{\bar v}$$
$$+q\lambda\Big(q\,\hat{M}_z\,\hat{\mathcal{D}}_v\,T_v^2\,T_-^2+\hat{\mathcal{D}}_-\,T_-$$
$$+\hat{M}_z\,\hat{M}_{\bar z}\,\hat{\mathcal{D}}_+\,(T_vT_-)^{-1}\,T_{\bar v}+q^{-1}\,\hat{M}_{\bar z}\,\hat{\mathcal{D}}_{\bar v}$$
$$-\lambda\,\hat{M}_v\,\hat{M}_{\bar z}\,\hat{\mathcal{D}}_-\,\hat{\mathcal{D}}_+\,T_{\bar v}\Big)\,\hat{\mathcal{D}}_{\bar z}\,\hat{\mathcal{D}}_z\,T_z\Big\}\,(T_vT_-T_+)^{-1}\,\hat\varphi=\hat\varphi' \qquad (71b)$$

where $\hat\varphi, \in C^\Lambda$ and satisfies (71a), $\hat\varphi' \in C^{\Lambda'}$, $\Lambda' = \Lambda - \alpha_{13}$, the corresponding signatures being as in (50). Clearly (71a, b) go into (50a, b) for $q = 1$. If we consider the irreducible factor-module L_Λ, which means that we should use (16) instead of (15), then we have a zero RHS in (71b) as in (50c).

4.3. Finally, we write down the q-difference analogues of the d'Alembert equation and of the equations ensuring its $U_q(sl(4))$ and $U_q(su(2,2))$ invariance and from which it follows (except for $a = 1$). Substituting (70) in (35) we obtain

$$R_3\,\hat\varphi=\hat{\mathcal{D}}_{\bar z}\,T_{\bar z}\,\hat\varphi=0 \qquad (72a)$$

$$\left([a-1]_q\,R_1^q\,R_2^q-[a]_q\,R_2^q\,R_1^q\right)\hat\varphi$$
$$=\Big(q^2\,[a-1]_q\,\left(q\,\hat{\mathcal{D}}_v\,T_v^2\,T_-^2+\hat{M}_{\bar z}\,\hat{\mathcal{D}}_+\,(T_vT_-)^{-1}\,T_{\bar v}\right)T_z$$
$$-q^{1-a}\,\Big(q\,\hat{M}_z\,\hat{\mathcal{D}}_v\,T_v^2\,T_-^2+\hat{\mathcal{D}}_-\,T_-$$
$$+\hat{M}_z\,\hat{M}_{\bar z}\,\hat{\mathcal{D}}_+\,(T_vT_-)^{-1}\,T_{\bar v}+q^{-1}\,\hat{M}_{\bar z}\,\hat{\mathcal{D}}_{\bar v}$$
$$-\lambda\,\hat{M}_v\,\hat{M}_{\bar z}\,\hat{\mathcal{D}}_-\,\hat{\mathcal{D}}_+\,T_{\bar v}\Big)\,\hat{\mathcal{D}}_z\Big)\,T_z\,(T_vT_-T_+)^{-1}\,\hat\varphi=0 \qquad (72b)$$

$$\left(R_3^q\,R_2^q-[2]_q\;\;R_2^q\,R_3^q\right)R_1^q\,R_2^q\,\hat\varphi=0\,. \qquad (72c)$$

As in the $q = 1$ case we use (72a) to split (72b) in two equations and to simplify (72c). Finally, we have

$$\hat{\mathcal{D}}_{\bar z}\,T_{\bar z}\,\hat\varphi=0 \qquad (73a)$$

$$\left(q^{a+2}\,[a-1]_q\,\hat{\mathcal{D}}_v\,T_z\,T_v^2\,T_--\left(q\,\hat{M}_z\,\hat{\mathcal{D}}_v\,T_v^2\,T_-+\hat{\mathcal{D}}_-\right)\hat{\mathcal{D}}_z\right)T_z\,(T_vT_+)^{-1}\,\hat\varphi=0 \qquad (73b)$$

$$\Big(q^{a+1}\,[a-1]_q\,\hat{\mathcal{D}}_+\,T_z\,(T_vT_-)^{-1}\,T_{\bar v}-\big(\hat{M}_z\,\hat{\mathcal{D}}_+\,(T_vT_-)^{-1}\,T_{\bar v}+q^{-1}\,\hat{\mathcal{D}}_{\bar v}$$
$$-\,\lambda\,\hat{M}_v\,\hat{\mathcal{D}}_-\,\hat{\mathcal{D}}_+\,T_{\bar v}\big)\,\hat{\mathcal{D}}_z\Big)\,T_z\,(T_vT_-T_+)^{-1}\,\hat\varphi=0 \qquad (73b')$$

$$\Big\{\big(\hat{\mathcal{D}}_v\,\hat{\mathcal{D}}_{\bar v}\,T_v^3\,T_{\bar v}^{-1}-q\,\hat{\mathcal{D}}_-\,\hat{\mathcal{D}}_+\,T_-^{-2}\big)T_z\,T_v^{-1}$$
$$-q\,\hat{\mathcal{D}}_-\,\hat{\mathcal{D}}_+\,\left(T_vT_z^{-1}-T_v^{-1}T_z\right)\Big\}\,(T_vT_+)^{-1}\,T_z\,T_-\,T_{\bar v}^2\;\;\hat\varphi=0\,. \qquad (73c)$$

In addition, if $a \in \mathbb{N}$ we also have

$$\left(\hat{\mathcal{D}}_z\ T_z\ (T_v T_- T_+ T_{\bar{v}})^{-1}\right)^a\ \hat{\varphi} = 0\,. \tag{73d}$$

In the scalar case $a = 1$ the relevant equations are (73*a, c, d*), in particular, using (73*c*) and adding a non-trivial RHS we obtain the conditionally $U_q(sl(4))$ and $U_q(su(2,2))$ invariant q-d'Alembert equation

$$\Big\{(\hat{\mathcal{D}}_v\ \hat{\mathcal{D}}_{\bar{v}}\ T_v^3\ T_{\bar{v}}^{-1} - q\ \hat{\mathcal{D}}_-\ \hat{\mathcal{D}}_+\ T_-^{-2})T_v^{-1}$$
$$-q\lambda\ \hat{\mathcal{D}}_-\ \hat{\mathcal{D}}_+\ \ \hat{M}_v\ \hat{\mathcal{D}}_v\Big\}\ (T_v T_+)^{-1}\ T_-\ T_{\bar{v}}^2\ \ \hat{\varphi} = \hat{\varphi}' \qquad a = 1\,. \tag{73e}$$

Analogously one may write down explicitly the conjugate invariant equations.

Clearly, for $q = 1$ (73*c, e*) go into the d'Alembert equations (57*d, c*), respectively.

Appendix A. Synopsis on $U_q(sl(4))$

The quantum algebra $U_q(sl(4))$ is defined as the associative algebra over $\mathbb{C}$ with Chevalley generators $X_j^{\pm}$, H_j , $j = 1, 2, 3$, and with the relations [8, 9]:

$$[H_j\ ,\ H_k] = 0 \qquad [H_j\ ,\ X_k^{\pm}] = \pm a_{jk} X_k^{\pm} \qquad [X_j^+\ ,\ X_k^-] = \delta_{jk}\ [H_j]_q \tag{A1a}$$

$$\begin{aligned}&\left(X_j^{\pm}\right)^2 X_k^{\pm} - [2]_q X_j^{\pm} X_k^{\pm} X_j^{\pm} + X_k^{\pm}\left(X_j^{\pm}\right)^2 = 0 \qquad (jk) = (12),\ (21),\ (23),\ (32)\\ &[X_1^{\pm}, X_3^{\pm}] = 0\end{aligned} \tag{A1b}$$

where $[x]_q = \left(q^x - q^{-x}\right)/\lambda$, $\lambda \equiv q - q^{-1}$, $(a_{jk}) = (2(\alpha_j, \alpha_k)/(\alpha_j, \alpha_j))$, $j, k = 1, 2, 3$, is the Cartan matrix of $sl(4)$; $\alpha_1, \alpha_2, \alpha_3$ are the simple roots; the non-zero products between the simple roots are: $(\alpha_j, \alpha_j) = 2$, $j = 1, 2, 3$, $(\alpha_1, \alpha_2) = (\alpha_2, \alpha_3) = -1$. The non-simple positive roots are : $\alpha_{12} = \alpha_1 + \alpha_2$, $\alpha_{23} = \alpha_2 + \alpha_3$, $\alpha_{13} = \alpha_1 + \alpha_2 + \alpha_3$. The elements H_j span the Cartan subalgebra $\mathcal{H}$, while the elements $X_j^{\pm}$ generate the subalgebras $\mathcal{G}^{\pm}$.

The Cartan–Weyl basis for the non-simple roots is given by [9, 10, 22]

$$X_{jk}^{\pm} = \pm q^{\mp 1/2}\left(q^{1/2} X_j^{\pm} X_k^{\pm} - q^{-1/2} X_k^{\pm} X_j^{\pm}\right) \qquad (jk) = (12),\ (23) \tag{A2a}$$

$$\begin{aligned}X_{13}^{\pm} &= \pm q^{\mp 1/2}\left(q^{1/2} X_1^{\pm} X_{23}^{\pm} - q^{-1/2} X_{23}^{\pm} X_1^{\pm}\right)\\ &= \pm q^{\mp 1/2}\left(q^{1/2} X_{12}^{\pm} X_3^{\pm} - q^{-1/2} X_3^{\pm} X_{12}^{\pm}\right).\end{aligned} \tag{A2b}$$

All other commutation relations for the generators follow from these definitions [22] ($X_{ii}^{\pm} \equiv X_i^{\pm}$):

$$[X_a^+\ ,\ X_{ab}^-] = -q^{H_a} X_{a+1b}^- \qquad [X_b^+\ ,\ X_{ab}^-] = X_{ab-1}^- q^{-H_b} \qquad 1 \leqslant a < b \leqslant 3 \tag{A3a}$$

$$[X_a^-\ ,\ X_{ab}^+] = X_{a+1b}^+ q^{-H_a} \qquad [X_b^-\ ,\ X_{ab}^+] = -q^{H_b} X_{ab-1}^+ \qquad 1 \leqslant a < b \leqslant 3 \tag{A3b}$$

$$X_a^{\pm} X_{ab}^{\pm} = q X_{ab}^{\pm} X_a^{\pm} \qquad X_b^{\pm} X_{ab}^{\pm} = q^{-1} X_{ab}^{\pm} X_b^{\pm} \qquad 1 \leqslant a < b \leqslant 3 \tag{A3c}$$

$$X_{12}^{\pm} X_{13}^{\pm} = q X_{13}^{\pm} X_{12}^{\pm} \qquad X_{23}^{\pm} X_{13}^{\pm} = q^{-1} X_{13}^{\pm} X_{23}^{\pm} \tag{A3d}$$

$$[X_2^\pm\,,\ X_{13}^\pm]=0 \qquad [X_2^\pm\,,\ X_{13}^\mp]=0 \tag{A3e}$$

$$[X_{12}^+\,,\ X_{13}^-]=-q^{2(H_1+H_2)}X_3^- \qquad [X_{12}^-\,,\ X_{13}^+]=X_3^+q^{-2(H_1+H_2)} \tag{A3f}$$

$$[X_{23}^+\,,\ X_{13}^-]=X_1^-q^{-2(H_2+H_3)} \qquad [X_{23}^-\,,\ X_{13}^+]=-q^{2(H_2+H_3)}X_1^+ \tag{A3g}$$

$$[X_{12}^\pm\,,\ X_{23}^\pm]=\lambda X_2^\pm X_{13}^\pm \qquad [X_{12}^\pm\,,\ X_{23}^\mp]=-\lambda q^{\pm H_2}X_1^\pm X_3^\mp\,. \tag{A3h}$$

Appendix B. Representation action of $U_q(sl(4))$ on the reduced functions

The representation (left) action of $U_q(sl(4))$ on the reduced functions $\hat\varphi$ of the representation space C^Λ, with signature $\chi=\chi(\Lambda)=(m_1,m_2,m_3)$ (see equations (65)), is given as follows [21, 19] ($k_i\equiv q^{H_i/2}$, $r_i=m_i-1$):

$$\hat\pi_\chi(k_1)\,\hat\varphi(\bar Y)=q^{-r_1/2}\,T_z\,(T_vT_+)^{1/2}\,(T_-T_{\bar v})^{-1/2}\,\hat\varphi(\bar Y) \tag{B1a}$$

$$\hat\pi_\chi(k_2)\,\hat\varphi(\bar Y)=q^{-r_2/2}\,T_-\,(T_vT_{\bar v})^{1/2}\,(T_zT_\eta)^{-1/2}\,\hat\varphi(\bar Y) \tag{B1b}$$

$$\hat\pi_\chi(k_3)\,\hat\varphi(\bar Y)=q^{-r_3/2}\,T_\eta\,(T_+T_{\bar v})^{1/2}\,(T_vT_-)^{-1/2}\,\hat\varphi(\bar Y) \tag{B1c}$$

$$\begin{aligned}\hat\pi_\chi(X_1^+)\,\hat\varphi(\bar Y)&=(1/\lambda)\,q^{-1}\,\hat M_z\,(T_-T_{\bar v})^{1/2}\,(T_vT_+)^{-1/2}\left(q^{r_1}T_z^{-1}-q^{-r_1}T_z\right)\hat\varphi(\bar Y)\\&\quad+q^{-r_1-1}\,\hat M_v\,\hat{\mathcal D}_-\,T_z\,(T_vT_{\bar v})^{1/2}\,(T_-T_+)^{-1/2}\,\hat\varphi(\bar Y)\\&\quad+q^{-r_1-1}\,\hat M_+\,\hat{\mathcal D}_{\bar v}\,T_z\,(T_vT_+)^{1/2}\,(T_-T_{\bar v})^{-1/2}\,\hat\varphi(\bar Y)\end{aligned} \tag{B2a}$$

$$\begin{aligned}\hat\pi_\chi(X_2^+)\,\hat\varphi(\bar Y)&=q^{r_2}\,\hat M_v\,\hat{\mathcal D}_z\,T_-^{-1}\,(T_zT_\eta)^{1/2}\,(T_vT_{\bar v})^{-1/2}\,\hat\varphi(\bar Y)\\&\quad+(1/\lambda)\,\hat M_-\,(T_zT_{\bar v})^{1/2}\,(T_vT_\eta)^{-1/2}\\&\quad\left(q^{-r_2}T_vT_-T_{\bar v}(T_zT_\eta)^{-1}-q^{r_2}T_zT_\eta(T_vT_-T_{\bar v})^{-1}\right)\hat\varphi(\bar Y)\\&\quad+q^{-r_2}\,\hat M_v\,\hat M_{\bar v}\,\hat{\mathcal D}_+\,(T_v\,T_-T_{\bar v}^3)^{1/2}\,(T_zT_\eta^3)^{-1/2}\,\hat\varphi(\bar Y)\\&\quad+q^{-r_2}\,\hat M_{\bar v}\,\hat{\mathcal D}_\eta\,T_-\,(T_vT_{\bar v})^{1/2}\,(T_zT_\eta)^{-1/2}\,\hat\varphi(\bar Y)\end{aligned} \tag{B2b}$$

$$\begin{aligned}\hat\pi_\chi(X_3^+)\,\hat\varphi(\bar Y)&=-q^{r_3-1}\,\hat M_+\,\hat{\mathcal D}_v\,T_\eta^{-1}\,(T_vT_-)^{1/2}\,(T_+T_{\bar v})^{-1/2}\,\hat\varphi(\bar Y)\\&\quad-q^{r_3-1}\,\hat M_{\bar v}\,\hat{\mathcal D}_-\,T_\eta^{-1}\,(T_v^3T_-)^{1/2}\,(T_+^3T_{\bar v})^{-1/2}\,\hat\varphi(\bar Y)\\&\quad+(1/\lambda)\,q^{-1}\,\hat M_\eta\,(T_+T_{\bar v})^{1/2}\,(T_vT_-)^{-1/2}\left(q^{-r_3}T_\eta-q^{r_3}T_\eta^{-1}\right)\hat\varphi(\bar Y)\end{aligned} \tag{B2c}$$

$$\begin{aligned}\hat\pi_\chi(X_1^-)\,\hat\varphi(\bar Y)&=q\,\hat{\mathcal D}_z\,(T_-T_{\bar v})^{1/2}\,(T_vT_+)^{-1/2}\,\hat\varphi(\bar Y)\\&\quad+q^2\,\hat M_-\,\hat{\mathcal D}_v\,T_z\,(T_-T_{\bar v})^{1/2}\,(T_vT_+)^{-1/2}\,\hat\varphi(\bar Y)\\&\quad+q^2\,\hat M_{\bar v}\,\hat{\mathcal D}_+\,T_z\,(T_vT_{\bar v})^{1/2}\,(T_-T_+)^{-1/2}\,\hat\varphi(\bar Y)\end{aligned} \tag{B3a}$$

$$\hat\pi_\chi(X_2^-)\,\hat\varphi(\bar Y)=-\,\hat{\mathcal D}_-\,(T_vT_\eta)^{1/2}\,(T_zT_{\bar v})^{-1/2}\,\hat\varphi(\bar Y) \tag{B3b}$$

$$\begin{aligned}\hat\pi_\chi(X_3^-)\,\hat\varphi(\bar Y)&=-\hat M_v\,\hat{\mathcal D}_+\,T_\eta^{-1}\,(T_+T_{\bar v}^3)^{1/2}\,(T_vT_-^3)^{-1/2}\,\hat\varphi(\bar Y)\\&\quad-\hat M_-\,\hat{\mathcal D}_{\bar v}\,T_\eta^{-1}\,(T_+T_{\bar v})^{1/2}\,(T_vT_-)^{-1/2}\,\hat\varphi(\bar Y)\\&\quad-q\,\hat{\mathcal D}_\eta\,(T_+T_{\bar v})^{1/2}\,(T_vT_-)^{-1/2}\,\hat\varphi(\bar Y)\,.\end{aligned} \tag{B3c}$$

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